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Boltzmann Equation on a Lattice: Existence and Uniqueness of Solutions

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Cercignani, Greenberg, and Zweifel proved the existence and uniqueness of solutions of the Boltzmann equation on a toroidal lattice under the assumption that the collision kernel is bounded. We give an alternative, considerably simpler, proof which is based on a fixed point argument.

KEY WORDS: Nonlinear Boltzmann equation; lattice approximation.

1. INTRODUCTION

The Boltzmann equation, which we write in the form

$$\frac{\partial}{\partial t} f(x, v, t) = -v \cdot \nabla_x f(x, v, t) + \int dv'\, dv''\, K(v|v', v'') f(x, v'', t) f(v, v', t) - \int dv''\, K(v'|v, v'') f(x, v'', t) f(x, v, t) \tag{1}$$

with a suitable transition kernel $dv\, K(v|v', v'')$, has two types of singularities: For a transition kernel coming from a finite-range potential, $\int dv\, K(v|v', v'')$ is unbounded as $|v' - v''|$. The general feeling is that this singularity can be controlled by suitable bounds on the moments of the initial datum. For the spatially homogeneous case this is precisely what has been proved.[1-3] The other singularity arises from the fact that a collision occurs at a certain point in space [formally the transition kernel contains the delta functions $\delta(x - x')\, \delta(x - x'')$]. This singularity is rather severe and up to now there has been no satisfactory way to deal with it: For short times (on the order of one-fifth of a mean free time) existence and uniqueness has been proved for a large class of initial data.[4-6] For initial data in some sense close to equilib-

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rium one has existence and uniqueness for all times and approach to equilibrium.[7,8]

Cercignani *et al.*[9] have proposed to put the Boltzmann equation on a lattice and to remove thereby the spatial singularity. Formally, (1) is modified in such a way that x runs over a toroidal lattice and ∇_x is replaced by the difference operator. The authors then show existence and uniqueness of the solutions under the assumption that K is bounded. Physically, the latter condition means that at high velocities in a collision, particles pass through each other without being deflected.

We want to consider here the same model, slightly generalized to arbitrary spatial domains and specularly reflecting boundary conditions. In particular, no attempt is made to remove either the cross section or the lattice cutoff.

Our, completely different, method of proof is based on a simple observation.[2] Let ϕ be the initial datum and consider the equation

$$\frac{\partial}{\partial t} f(x, v, t) = -v\cdot\nabla_x f(x, v, t) + \int dv'\, dv''\, K(v|v', v'')\phi(x, v'')f(x, v', t) - \int dv''\, K(v'|v, v'')\phi(x, v'')f(x, v, t) \tag{2a}$$

This is a linear Boltzmann equation (transport equation), which describes the motion of "test particles" with distribution $f(x, v, t)$ through a fluid characterized by the single-particle distribution $\phi(x, v)$. The test particles move freely and collide once in a while with a fluid particle. One solves (2a) for $0 \leqslant t \leqslant t_1$ with initial datum $f(x, v, 0) = \phi(x, v)$ and denotes this solution by $f_1(x, v, t)$. Since in the original Boltzmann equation any fluid particle could have been chosen as a test particle, the state of the fluid itself changes. This is taken into account by considering for the next time interval $t_1 \leqslant t \leqslant t_2$ the transport equation (2a) with a modified state of the fluid, i.e.,

$$\frac{\partial}{\partial t} f(x, v, t) = -v\cdot\nabla_x f(x, v, t) + \int dv'\, dv''\, K(v|v', v'')\phi_1(x, v'')f(x, v', t) - \int dv''\, K(v'|v, v'')\phi_1(x, v'')f(x, v, t) \tag{2b}$$

where $\phi_1(x, v) = f_1(x, v, t_1)$. One solves (2b) for $0 \leqslant t \leqslant t_2 - t_1$ with initial datum $f(x, v, 0) = \phi_1(x, v)$ and denotes this solution by $f_2(x, v, t)$, etc. Intuitively, $f_1(t)$ for $0 \leqslant t \leqslant t_1$, $f_2(t - t_1)$ for $t_1 \leqslant t \leqslant t_2$, $f_3(t - t_2)$ for $t_2 \leqslant t \leqslant t_3$,... will be an approximate solution of (1) with initial datum ϕ provided that $t_{j+1} - t_j$ is small.

[2] I learned from M. Aizenman to think about the Boltzmann equations in this way. Later we discovered that these ideas were already contained in a paper by McKean[10] under the notion of a nonlinear Markov process.

By reformulating the above argument in a continuous way, the solution of (1) is given as a fixed point of a certain mapping Z. Under our (physically rather restrictive) assumptions we prove then that Z has a unique fixed point.

2. FORMULATION OF THE PROBLEM

First, we construct a discrete version of $-v\cdot\nabla_x$. Let $\Lambda \subset Z^d$ be a simply connected domain. We consider the following random motion of a particle with phase space $\Lambda \times R^d$. If the particle is at the point $x \in \Lambda$ with velocity $v \in R^d$, $|v| \neq 0$, $v = (\epsilon_1|v_1|,..., \epsilon_d|v_d|)$, $\epsilon_j = \pm 1$, then it jumps to $x' = (x_1,..., x_j + \epsilon_j,..., x_d)$, $v' = v$, with probability $|v_j|/(|v_1|) + \cdots |v_d|)$. If $(x_1,..., x_j + \epsilon_j,..., x_d) \notin \Lambda$, then the particle jumps to $x' = x$, $v' = (v_1,..., -v_j,..., v_d)$ with the same probability. The jump rate is $|v_1| + \cdots + |v_d|$. If $v = 0$, then the particle stays at x with probability one. This defines a Markov jump process $x(t)$, $v(t)$. If $E_{x,v}$ denotes the expected value conditioned on the particle starting at x, v, then, for $\Lambda = Z^d$, $E_{x,v}(x(t)) = x + vt$, i.e., on the average the particle moves freely.

Let e^{At}, $t \geqslant 0$, be the forward Markovian semigroup of $x(t)$, $v(t)$, i.e.,

$$\sum_{x\in\Lambda} \int_{R^d} dv\, g(x, v)(e^{At}f)(x, v) = \sum_{x\in\Lambda} \int_{R^d} dv\, f(x, v) E_{x,v}(g(x(t), v(t))) \tag{3}$$

for all $g \in L^\infty(\Lambda \times R^d)$, $f \in L^1(\Lambda \times R^d)$. Then e^{At} is a strongly continuous, positivity- and norm-preserving semigroup on $L^1(\Lambda \times R^d)$. The domain $D(A)$ of A is $D(A) = \{f \in L^1(\Lambda \times R^d) |\ |v| f \in L^1(\Lambda \times R^d)\}$. It is straightforward to check that for a box $\Lambda \subset Z^d$ with periodic boundary conditions (i.e., on a torus) the above construction leads to the same A as in Ref. 9.

Let $K(dv|v', v'')$ be a transition kernel with the following properties:

(i)For each $v' \in R^d$, $v'' \in R^d$, $K(dv|v', v'')$ is a measure on R^d.

(ii) $(v', v'') \to \int_\Delta K(dv|v', v'')$ is measurable for every Borel set $\Delta \subset R^d$.

(iii) $\text{ess-sup}_{v',v''\in R^d} \int_{R^d} K(dv|v', v'') \leqslant c$.

(iv) $\int_{R^{2d}} dv'\, dv''\, K(dv|v', v'') f(v') g(v'')$ is absolutely continuous with respect to dv for all $f, g \in L^1(R^d)$.

For notational simplicity the dv-absolutely continuous measure

$$\int_{R^{2d}} dv'\, dv''\, K(dv|v', v'') f(v') g(v'')$$

is then identified with its density.

In passing, we note that for the Boltzmann equation corresponding to an interaction potential the kernel K is of the form

$$\int_\Delta K(dv|v', v'') = \int_\Delta dv\, \delta((v' - v)\cdot(v'' - v)) F(|v' - v|, |v' - v''|) \tag{4}$$

where the measurable function $F \geqslant 0$ can be computed from the differential cross section of the potential, e.g., $F = 1$ for hard spheres. Properties (i), (ii), and (iv) are valid and (iii) will hold for a large class of F's. However, for those F's coming from a potential, (iii) cannot be satisfied.

The Boltzmann equation on a lattice with cross-section cutoff is then

$$
\begin{aligned}
\frac{\partial}{\partial t} f(x, v, t) &= (Af)(x, v, t) + \int_{R^{2d}} dv'\, dv''\, K(dv|v', v'') f(x, v'', t) f(x, v', t) \\
&\quad - \int_{R^{2d}} dv''\, K(dv'|v, v'') f(x, v'', t) f(x, v, t) \\
&\equiv (Af)(x, v, t) + Q(f(t), f(t))(x, v)
\end{aligned}
\tag{5}
$$

We will also consider the integral version of (5)

$$
f(t) = e^{At}\phi + \int_0^t ds\, e^{A(t-s)} Q(f(s), f(s)) \tag{6}
$$

where $\phi = f(0)$ is the initial datum.

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Theorem 1. Let K satisfy (i)–(iv). Then for any initial datum $\phi \in L^1(\Lambda \times R^d)$, $\phi \geqslant 0$, (6) has a unique solution $f(t)$, $t \geqslant 0$, with $f(0) = \phi$. Furthermore, $f(t) \geqslant 0$ and $\|f(t)\| = \|\phi\|$.

Proof. Let us denote by $L_+{}^1(\Lambda \times R^d)$ the positive functions and by $L^1_{+,1}(\Lambda \times R^d)$ the positive functions normalized to one in $L^1(\Lambda \times R^d)$. The norm of $L^1(\Lambda \times R^d)$ is denoted by $\|\cdot\|$. Let $\phi \in L_+{}^1(\Lambda \times R^d)$. We define the operator $B^\psi: L^1(\Lambda \times R^d) \to L^1(\Lambda \times R^d)$ by

$$
\begin{aligned}
(B^\psi f)(x, v) &= \int_{R^{2d}} dv'\, dv''\, K(dv|v', v'') \psi(x, v'') f(x, v') \\
&\quad - \int_{R^{2d}} dv''\, K(dv'|v, v'') \psi(x, v'') f(x, v)
\end{aligned}
\tag{7}
$$

Again, by (iv), the first term is of the form $g(x, v)\, dv$ and is identified with $g(x, v)$.

Lemma 2. We have

$$
\|B^\psi\| \leqslant 2c\|\psi\|, \qquad \|B^{\psi_1} - B^{\psi_2}\| \leqslant 2c\|\psi_1 - \psi_2\| \tag{8}
$$

If $t \mapsto \psi(t) \in L^1_{+,1}(\Lambda \times R^d)$ is continuous, then $t \mapsto B^{\psi(t)}$ is norm continuous and $\|B^{\psi(t)}\| \leqslant 2c$ for all t.

Proof. We have

$$
\begin{aligned}
&\|B^{\psi_1}f - B^{\psi_2}f\| \\
&\quad \leqslant \sum_{x\in\Lambda} \int_{R^{3d}} dv'\, dv''\, K(dv|v', v'')|\psi_1(x, v'') - \psi_2(x, v'')|\, |f(x, v')| \\
&\qquad + \sum_{x\in\Lambda} \int_{R^{3d}} dv\, dv''\, K(dv'|v, v'')|\psi_1(x, v'') - \psi_2(x, v'')|\, |f(x, v)| \\
&\quad \leqslant 2 \sum_{x\in\Lambda} \int_{R^d} dv''\, |\psi_1(x, v'') - \psi_2(x, v'')| \int_{R^d} dv'\, |f(x, v')| \\
&\qquad \times \operatorname{ess-sup} \int_{R^d} K(dv|v', v'') \\
&\quad \leqslant 2c\|\psi_1 - \psi_2\|\, \|f\| \quad \blacksquare
\end{aligned}
$$

Let $\phi \in L^1_{+,1}(\Lambda \times R^d)$ be the initial datum and let $[0, T] \ni t \mapsto \psi(t) \in L^1_{+,1}(\Lambda \times R^d)$ be continuous, $\psi(0) = \phi$. We consider the equation

$$\frac{d}{dt} f(t) = Af(t) + B^{\psi(t)}f(t) \tag{9}$$

Equation (9) has the structure of a transport equation with time-dependent collision term $B^{\psi(t)}$. Since $t \mapsto B^{\psi(t)}$ is uniformly bounded and norm-continuous, by standard time-dependent perturbation theory,[11] the mild solution of (9) is the two-parameter family of contractions $U^{\psi(\cdot)}(s, t)$, $0 \leqslant s \leqslant t \leqslant T$, which preserve positivity and norm and satisfy

$$U^{\psi(\cdot)}(t_1, t_2)U^{\psi(\cdot)}(t_2, t_3) = U^{\psi(\cdot)}(t_1, t_3), \qquad U^{\psi(\cdot)}(t, t) = 1$$

Furthermore $U^{\psi(\cdot)}(s, t)$ is jointly strongly continuous in s and t.

Let $X = \{\psi(\cdot)|[0, T] \ni t \mapsto \psi(t) \in L^1_{+,1}(\Lambda \times R^d),\ \psi(\cdot)$ is continuous, $\psi(0) = \phi\}$ and define the metric m on X by

$$m(\psi_1(\cdot), \psi_2(\cdot)) = \sup_{t\in[0,T]} \|\psi_1(t) - \psi_2(t)\|$$

(X, m) is a complete metric space. We define a mapping $Z\colon X \to X$ by

$$Z\colon \quad \psi(t) \to U^{\psi(\cdot)}(0, t)\phi, \qquad 0 \leqslant t \leqslant T \tag{10}$$

Clearly, a solution of (6) is a fixed point of Z. Therefore we only have to show that Z is a strict contraction for T small enough. Then by the contraction mapping principle Z has a unique fixed point. Since $\phi \in L^1_{+,1}(\Lambda \times R^d)$ was arbitrary and since $U^{\psi(\cdot)}(0, t)$ preserves positivity and norm, by iteration we obtain a solution for all $t \geqslant 0$.

Lemma 3. We have

$$m(Z\psi_1(\cdot), Z\psi_2(\cdot)) \leqslant (e^{2cT} - 1)m(\psi_1(\cdot), \psi_2(\cdot)) \tag{11}$$

Proof.

$$\begin{aligned}
&\| U^{\psi_1(\cdot)}(0, t)\phi - U^{\psi_2(\cdot)}(0, t)\phi \| \\
&= \left\| \int_0^t ds\, e^{A(t-s)}[B^{\psi_1(s)}U^{\psi_1(\cdot)}(0, s) - B^{\psi_2(s)}U^{\psi_2(\cdot)}(0, s)]\phi \right\| \\
&\leqslant \int_0^t ds\, \| B^{\psi_1(s)}U^{\psi_1(\cdot)}(0, s) - B^{\psi_2(s)}U^{\psi_1(\cdot)}(0, s) \| \\
&\quad + \| B^{\psi_2(s)}[U^{\psi_1(\cdot)}(0, s) - U^{\psi_2(\cdot)}(0, s)\phi \| \\
&\leqslant 2ct \sup_{s\in[0,t]} \| \psi_1(s) - \psi_2(s) \| \\
&\quad + 2c \int_0^t ds\, \| U^{\psi_1(\cdot)}(0, s)\phi - U^{\psi_2(\cdot)}(0, s)\phi \|
\end{aligned}$$

where we used $\|e^{At}\| = 1$, $\|U^{\psi_1(\cdot)}(0, s)\| = 1$, and Lemma 2. By Gronwall's lemma we obtain (11). ■

To obtain a solution of (5) one has to show the differentiability of $f(t)$. We use the following specialization of a more general result due to Voigt.[12]

Theorem 4 (Voigt). Let X be a Banach space and $V(t)$, $t \geqslant 0$, be a holomorphic semigroup of operators on X with generator A. Let F: $X \to X$ be a continuous and locally Lipschitz, i.e., for $x \in X$ there exists a neighborhood $\mathcal{O} \subset X$ of x and a constant L such that

$$\|F(y) - F(z)\| \leqslant L\|y - z\|$$

for all $y, z \in \mathcal{O}$. Let $\mu_0 \in X$. Let $[0, \infty) \ni t \mapsto \mu(t) \in X$ be continuous and satisfy the integral equation

$$\mu(t) = V(t)\mu_0 + \int_0^t V(t - s)F(\mu(s))\, ds$$

Then $\mu(t) \in D(A)$ for all $t > 0$ and $\mu(t)$ satisfies

$$\frac{d}{dt}\mu(t) = A\mu(t) + F(\mu(t)), \qquad 0 < t < \infty, \qquad \mu(0) = \mu_0$$

In our case, $f \mapsto Q(f, f)$ is clearly locally Lipschitz. One only has to show that e^{At} is a holomorphic semigroup.

Lemma 5. If $|\Lambda| < \infty$, then e^{At}, $t \geqslant 0$, is a bounded holomorphic semigroup.

Proof. Let $d = 1$. Then $\Lambda = [M, N]$. Now, $[M, N] \times \{|v|, -|v|\}$ is invariant. We have explicitly

$$\left.\begin{aligned} (Af)(x, v) &= |v|[f(x-1, v) - f(x, v)], & M+1 \leqslant x \leqslant N \\ (Af)(x, v) &= |v|[f(x, -v) - f(x, v)], & x = M \end{aligned}\right\} v > 0$$

$$\left.\begin{aligned} (Af)(x, v) &= |v|[f(x+1, v) - f(x, v)], & M \leqslant x \leqslant N-1 \\ (Af)(x, v) &= |v|[f(x, -v) - f(x, v)], & x = N \end{aligned}\right\} v < 0$$

By Fourier-transforming one checks that the spectrum of A lies in the sector $z\{| \, |\arg z - \pi| \leqslant \theta\}$, with

$$\tan \theta = \left(\sin \frac{\pi}{N-M+1}\right)\left(1 - \cos \frac{\pi}{N-M+1}\right)^{-1}$$

i.e., for $N - M$ large, $\theta \simeq \frac{1}{2}\pi - \pi/(N - M + 1)$. Since θ is independent of v, then A on $L^1([M, N] \times R)$ generates a holomorphic semigroup. For $d > 1$ one only has to consider the invariant subset

$$\Lambda \times \{(\epsilon_1|v_1|, \ldots, \epsilon_d|v_d|) | \epsilon_j = \pm 1, j = 1, \ldots, d\}. \quad \blacksquare$$

Therefore we obtain the following result:

Theorem 6. Let $|\Lambda| < \infty$ and K satisfy (i)–(iv). Then for any initial datum $\phi \in L^1(\Lambda \times R^d)$, $\phi \geqslant 0$, there exists a unique solution $f(t)$ with $f(0) = \phi$ of (5) for $t > 0$. Furthermore, $f(t) \geqslant 0$ and $\|f(t)\| = \|\phi\|$.

We have no comparable result for $|\Lambda| = \infty$.

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